Lect25-0419 Pi-1

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Given spaces X, Y

$$\mathcal{C}(X,Y) = \left\{ \text{all continuous mappings } X \longrightarrow Y \right\}$$

For $f, g \in \mathcal{C}(X,Y)$, $f \simeq g$ if

$\exists$ continuous $H : X \times [0,1] \longrightarrow Y$

$$\left.\begin{aligned} f(x) &= H(x,0) \\ g(x) &= H(x,1) \end{aligned}\right\} \forall\, x \in X$$

$$[X,Y] = \mathcal{C}(X,Y)/\simeq$$

Examples

① $Y = \mathbb{R}^n$, $n \geq 1$; or star-shaped

$X =$ any space

$c : X \longrightarrow Y$, $c(x) = 0 \in \mathbb{R}^n$ constant

$[X, \mathbb{R}^n] =$ singleton $= \{[c]\}$

Every $f : X \to \mathbb{R}$, $f \simeq c$

② $X = S^1$, $Y = \mathbb{C} \setminus \{0\} = \mathbb{R}^2 \setminus \{(0,0)\}$

$f \in \mathcal{C}(S^1, \mathbb{C}\setminus\{0\}) \longmapsto$ Invariant $\in \mathbb{Z}$

winding number

$$\#[S^1, \mathbb{C}\setminus\{0\}] \geq \#\mathbb{Z}$$

$$\#$$

$$\#[S^1, \mathbb{R}^n] = 1$$

Expect: $\mathbb{C}\setminus\{0\} \not\cong \mathbb{R}^n$

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Theorem you need!

* If $Y_1 = Y_2$ (homeomorphic)
  then $\forall$ space X, $[X, Y_1]$, $[X, Y_2]$ are bijective
* If $X_1 = X_2$ (homeomorphic)
  then $\forall$ space Y, $[X_1, Y]$, $[X_2, Y]$ are bijective

Idea of proof

Let $\varphi : Y_1 \longrightarrow Y_2$ be a homeomorphism

Define $\varphi_\# : [X, Y_1] \longrightarrow [X, Y_2]$ by

$$\underset{: X \to Y_1}{[f]} \longmapsto \underset{: X \to Y_1 \to Y_2}{[\varphi \circ f]}$$

Qu. What do we need to check?

(i) $\varphi_\#$ is well-defined, i.e. $[f] = [g] \Rightarrow [\varphi f] = [\varphi g]$

(ii) 1-1 $\}$ Hope: $(\varphi_\#)^{-1} = (\varphi^{-1})_\#$

(iii) onto

More precisely

(i) $[f] = [g] \Rightarrow [\varphi \circ f] = [\varphi \circ g]$

$$\underset{h_t}{f \simeq g} \qquad \underset{\varphi \circ h_t}{\varphi \circ f \simeq \varphi \circ g}$$

(ii), (iii) $\}$ $(\varphi_\#)^{-1} = (\varphi^{-1})_\#$

$$[f] \xrightarrow{\varphi_\#} [\varphi \circ f] \xrightarrow{(\varphi^{-1})_\#} [\varphi^{-1} \circ \varphi \circ f]$$

More general,

$$\begin{array}{ccccc} [f] & \xrightarrow{\varphi_\#} & [\varphi \circ f] & \xrightarrow{\psi_\#} & [\psi \circ \varphi \circ f] \\ \| & & & & \| \\ [g] & \longrightarrow & [\varphi \circ g] & \longrightarrow & [\psi \circ \varphi \circ g] \end{array}$$

The crucial argument used in (i), (ii), (iii) is the result below.

Ultimate Theorem Let X, Y, Z be spaces and

$$\begin{tikzcd} X \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & Y \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] & Z \end{tikzcd}$$

If $f_0 \overset{F}{\simeq} f_1$ and $g_0 \overset{G}{\simeq} g_1$

then $g_0 \circ f_0 \overset{H}{\simeq} g_1 f_1 \simeq g_1 f_0 \simeq g_0 f_1 : X \longrightarrow Z$

Proof

Construct $H : X \times [0,1] \longrightarrow Z$ by

$$H(x,t) = G(F(x,t), t)$$

The other homotopies are similar

Mapping of a pair

Let $A \subset X$, $B \subset Y$. Denote $f : (X, A) \longrightarrow (Y, B)$ meaning $f : X \longrightarrow Y$ and $f(A) \subset B$

Loop. Let X be a space with $x_0 \in X$

A loop in X based at x_0 is a continuous

$$\gamma : ([a,b], \{a,b\}) \longrightarrow (X, x_0)$$

i.e. $\gamma(a) = \gamma(b) = x_0$

A path begins and ends at x_0

Alternatively, may study

$$\gamma : (S^1, 1) \longrightarrow (X, x_0)$$

Loop htpy

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Given two loops $\gamma_0, \gamma_1 : [0,1] \longrightarrow X$ at x_0, they are loop homotopic if there exists a loop homotopy $L : [0,1] \times [0,1] \longrightarrow X$ such that

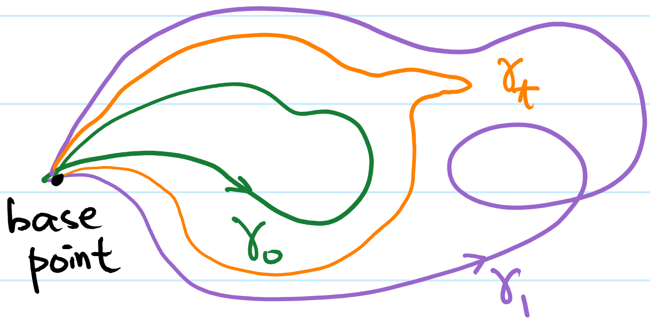
loop parameter homotopy time

$$\left.\begin{aligned} L(s,0) &= \gamma_0(s) \\ L(s,1) &= \gamma_1(s) \end{aligned}\right\} \quad s \in [0,1]$$

normal homotopy

$$L(0,t) = L(1,t) = x_0 \quad \forall \ t \in [0,1]$$

at any time t, it is a loop based at x_0



For $f, g : (X, A) \longrightarrow (Y, B)$ with $f|_A \equiv g|_A$

A homotopy rel A is a continuous mapping

$$H : X \times [0,1] \longrightarrow Y$$

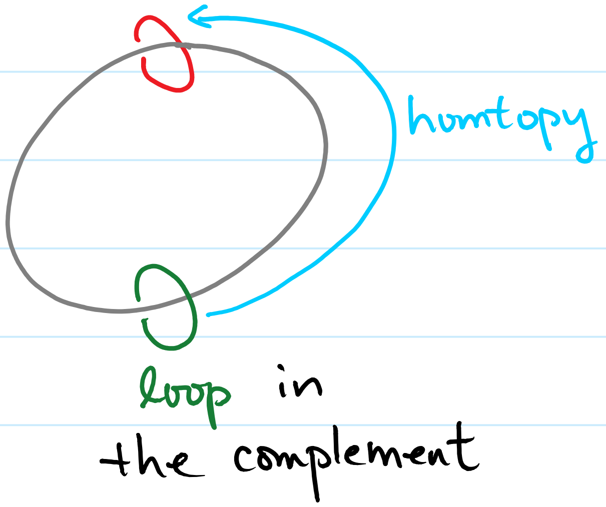
$$H(x,0) = f(x), \quad H(x,1) = g(x), \quad x \in X$$

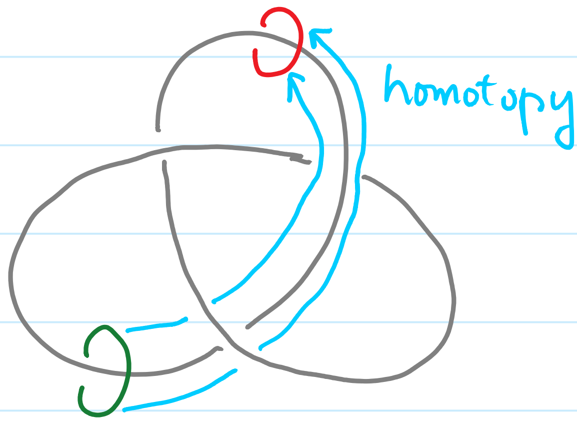
$$H(x,t) = f(x) = g(x), \quad x \in A, \quad t \in [0,1]$$

Example of importance of base point

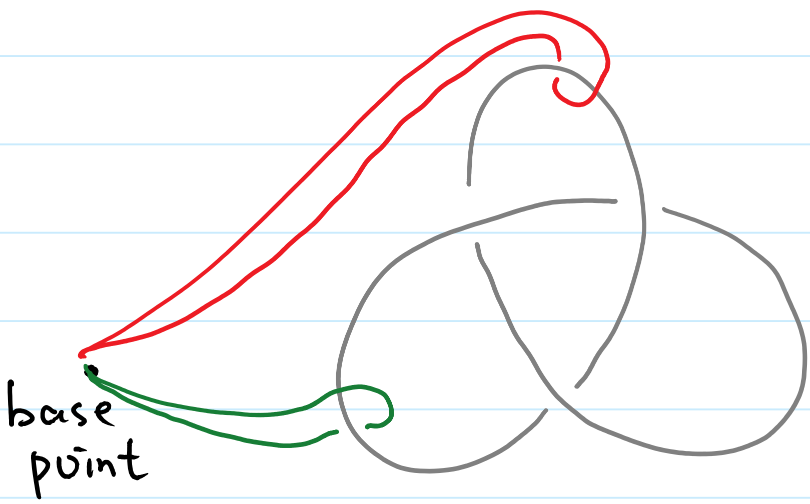
In topology, we often need to study complement of a knot

$\mathbb{R}^3 \setminus$ circle or $\mathbb{R}^3 \setminus$ Trefoil





If the loops have no base point, both $\mathbb{R}^3 \setminus$ circle, $\mathbb{R}^3 \setminus$ Trefoil has this homotopy class

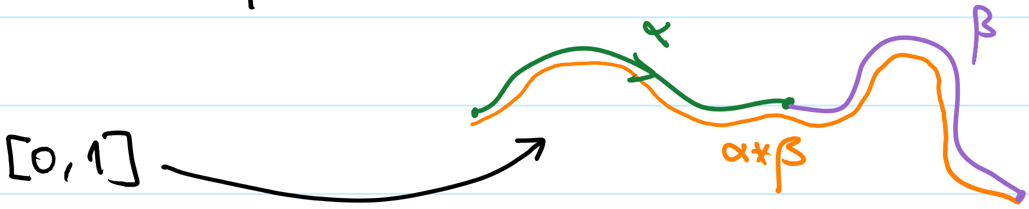


These two loops with base point are not homotopic. Therefore

$$\mathbb{R}^3 \setminus \text{circle} \neq \mathbb{R}^3 \setminus \text{Trefoil}$$

Concatenation Let $\alpha, \beta : [0,1] \to X$ be paths such that $\alpha(1) = \beta(0)$, define a path
$\alpha * \beta : [0,1] \longrightarrow X$ by

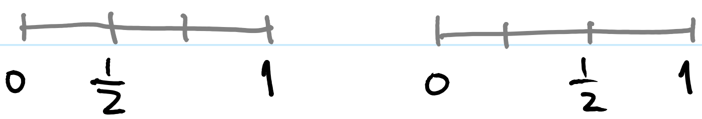
$$\alpha * \beta(s) = \begin{cases} \alpha(2s) & \text{if } s \in [0, \frac{1}{2}] \\ \beta(2s-1) & \text{if } s \in [\frac{1}{2}, 1] \end{cases}$$



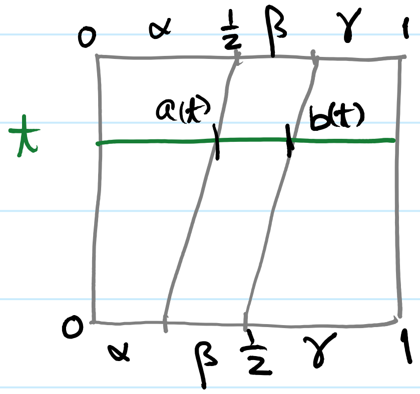
Note that as mappings from $[0,1] \longrightarrow X$

$$\alpha * (\beta * \gamma) \neq (\alpha * \beta) * \gamma$$

parameters



Proposition $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$ rel $\{0,1\}$



The homotopy needed

$$H(s,t) = \begin{cases} \alpha(\cdots) & s \in [0, a(t)] \\ \beta(\cdots) & s \in [a(t), b(t)] \\ \gamma(\cdots) & s \in [b(t), 1] \end{cases}$$

so that the whole α, β, γ are travelled.

By high school math,

$$a(t) = \frac{1}{4}(1-t) + \frac{1}{2}t = \frac{1}{4} + \frac{t}{4}, \quad b(t) = \frac{1}{2} + \frac{t}{4}$$

$$\alpha\left(\frac{s}{a(t)}\right), \quad \beta\left(\frac{s - a(t)}{b(t) - a(t)}\right), \quad \gamma\left(\frac{s - b(t)}{1 - b(t)}\right)$$

Group structure

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Knowing that associativity is true up to homotopy we need to check well-define up to homotopy!

Proposition $\alpha_0 \simeq \alpha_1$ and $\beta_0 \simeq \beta_1$, perhaps rel $\{0,1\}$

Then $\alpha_0 * \beta_0 \simeq \alpha_1 * \beta_1$, perhaps rel $\{0,1\}$

Result. $[\alpha]\cdot[\beta] = [\alpha * \beta]$ is well-defined

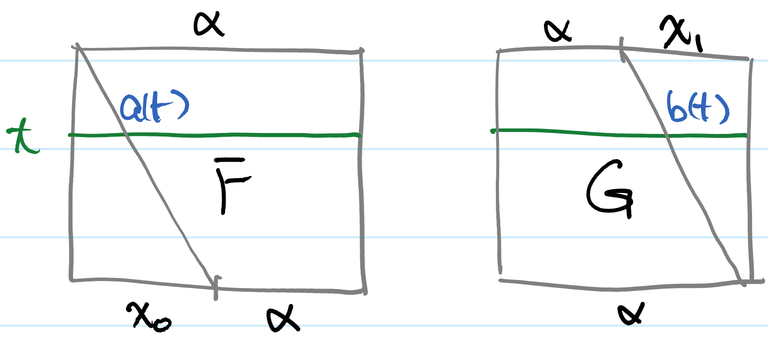
and $([\alpha]\cdot[\beta])\cdot[\gamma] = [\alpha]\cdot([\beta]\cdot[\gamma])$

Existence of Identity

Let $\alpha : [0,1] \longrightarrow X$, $\alpha(0) = x_0$, $\alpha(1) = x_1$

$c_0 : [0,1] \longrightarrow \{x_0\}$, $c_1 : [0,1] \longrightarrow \{x_1\}$

Then $c_0 * \alpha \simeq \alpha \simeq \alpha * c_1$ rel $\{0,1\}$



$$F(s,t) = \begin{cases} x_0 & s \in [0, a(t)] \\ \alpha\left(\dfrac{s - a(t)}{1 - a(t)}\right) & s \in [a(t), 1] \end{cases} \qquad a(t) = \frac{1-t}{2}$$

$$G(s,t) = \begin{cases} \alpha\left(\dfrac{s}{b(t)}\right) & s \in [0, b(t)] \\ x_1 & s \in [b(t), 1] \end{cases} \qquad b(t) = 1 - \frac{t}{2}$$

Existence of "Inverse"

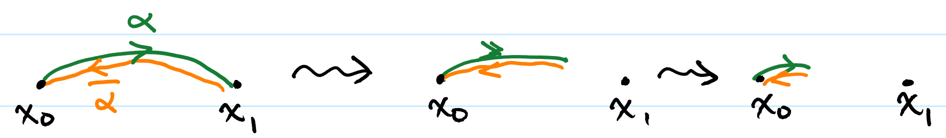
Let $\alpha : [0,1] \longrightarrow X$, $\alpha(0) = x_0$, $\alpha(1) = x_1$

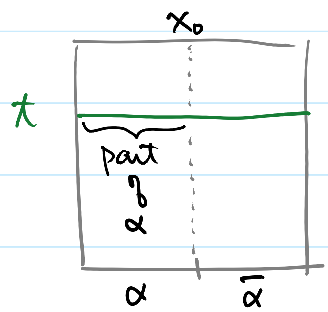
Define $\bar{\alpha} : [0,1] \longrightarrow X$ by

$$\bar{\alpha}(s) = \alpha(1-s)$$ travelling backward

Then $\bar{\alpha}(0) = x_1$, $\bar{\alpha}(1) = x_0$

Proposition $\alpha * \bar{\alpha} \simeq c_0$, $\bar{\alpha} * \alpha \simeq c_1$ rel $\{0,1\}$





$$H(s,t) = \begin{cases} \alpha(2s(1-t)) & s \in [0, \tfrac{1}{2}] \\ \bar{\alpha}(\text{exercise}) & s \in [\tfrac{1}{2}, 1] \end{cases}$$

Fundamental Group $\pi_1(X, x_0)$

(i) Set of loop homotopy classes $[\alpha]$, where

$$\alpha : ([0,1], \{0,1\}) \longrightarrow (X, x_0)$$

(ii) $\alpha \simeq \beta$ rel $\{0,1\}$

(iii) $[\alpha] \cdot [\beta] = [\alpha * \beta]$

(iv) $1 = [c_0]$

(v) $[\alpha]^{-1} = [\bar{\alpha}]$

Fund'l group

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Given a space X and a base point $x_0 \in X$

$\pi_1(X, x_0)$ = Fundamental Group of X at x_0

$\cup\!\!\!\!\shortmid$ $[\alpha]$ loop homotopy class

$\alpha : ([0,1], \{0,1\}) \longrightarrow (X, x_0)$

under homotopy rel $\{0,1\}$

Group structure

$$[\alpha]\cdot[\beta] = [\alpha * \beta]$$

$$1 = [c] \quad \text{where } c : X \longrightarrow \{x_0\}$$

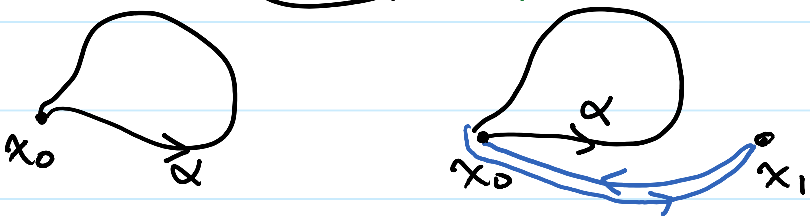
$$[\alpha]^{-1} = [\bar{\alpha}] \qquad \bar{\alpha}(s) = \alpha(1-s)$$

Independent of base point

If X is path connected and $x_0, x_1 \in X$

then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic

$\cup\!\!\!\!\shortmid$ Loops at x_0 $\longrightarrow$ $\cup\!\!\!\!\shortmid$ Loops at x_1



Let $\gamma : [0,1] \longrightarrow X$ be a path with

$\gamma(0) = x_1$, $\gamma(1) = x_0$, from x_1 to x_0

Then $\bar{\gamma}(s) = \gamma(1-s)$ is from x_0 to x_1

Define $\varphi : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ by

$$[\alpha] \longmapsto [\gamma * \alpha * \bar{\gamma}]$$

Invariant

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Need to check

① Well-defined: $\alpha_0 \simeq \alpha_1$ rel $\{0,1\}$

$$\Rightarrow \gamma * \alpha_0 * \bar{\gamma} \simeq \gamma * \alpha_1 * \bar{\gamma} \text{ rel } \{0,1\}$$

proved similarly in $[\alpha][\beta] = [\alpha * \beta]$

② Bijection: Obvious inverse by $\bar{\gamma}$ from x_0 to x_1

③ homomorphism: for $[\alpha], [\beta] \in \pi_1(X, x_0)$

$$(\varphi[\alpha]) \cdot (\varphi[\beta]) = \varphi([\alpha][\beta])$$

$$(\gamma * \alpha * \underbrace{\bar{\gamma}) * (\gamma}_{} * \beta * \bar{\gamma}) \qquad \gamma * (\alpha * \beta) * \bar{\gamma}$$

Clearly $\bar{\gamma} * \gamma \simeq c$

Qu. For $[\alpha] \overset{\varphi}{\longmapsto} [\gamma * \alpha * \bar{\gamma}]$, is φ independent of choice of γ from x_1 to x_0?

Theorem If $X \simeq Y$ are path connected then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$

The proof is a routine exercise of homotopy.

Simply connected A path connected space X is 1-connected if $\pi_1(X, x_0)$ is trivial

Examples.

(1) X is contractible $\Rightarrow$ X is 1-connected

(2) S^2, in general, S^n, $n \geq 2$ is 1-connected.

But they are not contractible